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College of Science for Women
Department of Mathematics

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Lecture
In linear algebra
Systems of Linear Equations and Matrices

References :

linear algebra and its applications . Third edition update . David c.lay .
2 - elementary linear algebra with applications, ninth edition , Bernard kolman , david hill.

Systems of Linear Equations and Matrices

Introduction

- **Why matrices?**
  - Information in science and mathematics is often organized into rows and columns to form regular arrays, called matrices.
- **What are matrices?**
  - tables of numerical data that arise from physical observations
- **Why should we need to learn matrices?**
  - because computers are well suited for manipulating arrays of numerical information
  - besides, matrices are mathematical objects in their own right, and there is a rich and important theory associated with them that has a wide variety of applications

Introduction to Systems of Linear Equations

- **Linear Equations**
  - **Any straight line in the *xy*-plane can be represented algebraically by the equation of the form**

    $$a_1 x + a_2 y = b$$

    an equation of this form is called a linear equation in the variables of *x* and *y*.
  - **generalization: linear equation in *n* variables**

    $$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

  - **the variables in a linear equation are sometimes called** *unknowns*
  - **examples of linear equations**

    $$x + 3y = 7,\ y = (1/2)x + 3z + 1, \text{ and } x_1 - 2x_2 - 3x_3 + x_4 = 7$$

  - **examples of non-linear equations**

    $$x + 3\sqrt{y} = 5, 2x + 2y - z + xz = 4, \text{ and } y = \sin x$$

Introduction to Systems of Linear Equations

- **Solution of a Linear Equation**
  - **A solution of a linear equation $a_1x_1+a_2x_2+...+a_nx_n=b$ is a sequence of *n* numbers $s_1, s_2,...,s_n$ such that the equation is satisfied when we substitute $x_1=s_1, x_2=s_2,...,x_n=s_n$.**
- **Solution Set**
  - **The set of solutions of the equation is called its** *solution set*, **or** *general solution*.
  - **Example:**

    Find the solution set of (a) $4x-2y=1$, and (b) $x_1-4x_2+7x_3=5$.
- **Linear Systems**
  - **A finite set of linear equations in the variables $x_1, x_2,...,x_n$ is called a** *system of linear equations* **or a** *linear system*.
  - **A sequence of numbers $s_1, s_2,...,s_n$ is a solution of the system if $x_1=s_1, x_2=s_2,...,x_n=s_n$ is a solution of every equation in the system.**

Introduction to Systems of Linear Equations

- **Not all linear systems have solutions.**
- **A linear system that has no solutions is said to be** *inconsistent*; **if there is at least one solution of the system, it is called** *consistent*.
- ***Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.***
- **An arbitrary system of** ***m*** **linear equations in** ***n*** **unknown can be written as**

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

- **The double subscripting of the unknown is a useful device that is used to specify the location of the coefficients in the system.**

Introduction to Systems of Linear Equations

- **Augmented Matrices**
  - **A system of *m* linear equations in *n* unknown can be abbreviated by writing only the rectangle array of numbers:**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This is called the *augmented matrix* **of the system.**

  - **The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but which is easier to solve.**

1. **Multiply an equation through by a nonlinear constant**
2. **Interchange two equations.**
3. **Add a multiple of one equation to another.**

Introduction to Systems of Linear Equations

- **Elementary Row Operations**
  - **Since the rows of an augmented matrix correspond to the equations in the associated system, the three operations above correspond to the following** *elementary row operations* **on the rows of the augmented matrix:**
    1. **Multiply a row through by a nonzero constant.**
    2. **Interchange two rows.**
    3. **Add a multiple of one row to another row.**
  - **Example: Using elementary row operations to solve the linear system**

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Gaussian Elimination

- **Echelon Forms**
  - **A matrix with the following properties is in** *reduced row-echelon form***:**
    1. **If a row does not consist entirely of zeros, then the first nonzero number in the row is 1. We call this a** *leading 1***.**
    2. **If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.**
    3. **In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.**
    4. **Each column that contains a leading 1 has zeros everywhere else.**
  - **A matrix that has the first three properties is said to be in** *row-echelon form***.**
  - **Example 1**

$$\begin{bmatrix}1&0&0&4\\0&1&0&7\\0&0&1&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&-2&0&1\\0&0&0&1&3\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}\quad\begin{bmatrix}1&4&-3&7\\0&1&6&2\\0&0&1&5\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&2&6&0\\0&0&1&-1&0\\0&0&0&0&1\end{bmatrix}$$

Gaussian Elimination

- A matrix in row-echelon form has zeros below each leading 1, whereas a matrix in reduced row-echelon form has zeros below and above each leading 1.
- The solution of a linear system may be easily obtained by transforming its augmented matrix to reduced row-echelon form.
- Example: Solutions of Four Linear Systems

$$(a)\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad (b)\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

$$(c)\begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (d)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- leading variables vs. free variables

Gaussian Elimination

- **Elimination Methods**
  - **to reduce any matrix to its reduced row-echelon form**
    1. **Locate the leftmost column that does not consist entirely of zeros.**
    2. **Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.**
    3. **If the entry that is now at the top of the column found in Step 1 is *a*, multiply the first row by 1/*a* in order to produce a leading 1.**
    4. **Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.**
    5. **Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the *entire* matrix is in row-echelon form.**
    6. **Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.**
  - **The first five steps produce a row-echelon form and is called** *Gaussian elimination*. **All six steps produce a reduced row-echelon form and is called** *Gauss-Jordan elimination*.

Gaussian Elimination

- **Example: Gauss-Jordan Elimination**

  Solve by Gauss-Jordan elimination.

$$
\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}
$$

- **Back-Substitution**
  - **to solve a linear system from its row-echelon form rather than reduced row-echelon form**
    1. **Solve the equations for the leading variables.**
    2. **Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.**
    3. **Assign arbitrary values to the free variables, if any.**
       - **Such arbitrary values are often called *parameters*.**

Gaussian Elimination

–Example: Gaussian Elimination

Solve

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 9 \\ 2x & + & 4y & - & 3z & = & 1 \\ 3x & + & 6y & - & 5z & = & 0 \end{array}$$

by Gaussian elimination and back-substitution.

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \xrightarrow{\text{Row-echelon}} \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The system corresponding to matrix yields

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 9 \\ & & y & - & \frac{7}{2}z & = & -\frac{17}{2} \\ & & & & z & = & 3 \end{array}$$

Solving the leading variables

$$\begin{array}{lccccc} x = & 9 & - & y & - & 2z \\ y = & -\frac{17}{2} & + & \frac{7}{2}z & & \\ z = & 3 & & & & \end{array}$$

$$z = 3 - y$$

$$y = 2$$

$$z = 3$$

- **Homogeneous Linear Systems**
  - **A system of linear equations is said to be** *homogeneous* **if the constant terms are all zero.**
  - **Every homogeneous system is consistent, since all system have $x_1=0$, $x_2=0,\ldots,x_n=0$ as a solution. This solution is called the** *trivial* **solution; if there are other solutions, they are called** *nontrivial solutions***.**
  - **A homogeneous system has either only the trivial solution, or has infinitely many solutions.**

Gaussian Elimination

Example: Gauss-Jordan Elimination Solve the following homogeneous system of linear equations by using Gauss-Jordan Elimination.

$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & 2x_2 & - & x_3 & & & + & x_5 & = & 0 \\ -x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0 \\ & & & & x_3 & + & x_4 & + & x_5 & = & 0 \end{array}$$

The augmented matrix for the system is

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reducing this matrix to reduced roe-echelon form, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equation is

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & & & + & x_5 = 0 \\ & & & & x_3 & + & x_5 = 0 \\ & & & & & x_4 & = 0 \end{array}$$

Gaussian Elimination

Solving for the leading variables yields

$$x_1 = -x_2 - x_5$$
$$x_3 = -x_5$$
$$x_4 = 0$$

Thus the general solution is

$$x_1 = -s - t$$

$$x_2 = s$$

$$x_3 = -t, \quad x_4 = 0, \quad x_5 = t$$

Note that the trivial solution is obtained when s=t=0

Matrices and Matrix Operations

- **Matrix Notation and Terminology**
  - **A *matrix* is a rectangular array of numbers. The numbers in the array are called the *entries* in the matrix.**
    - **Examples:**

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \quad [2 \quad 1 \quad 0 \quad -3] \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad [4]$$

  - **The size of a matrix is described in terms of the number of rows and columns it contains.**
    - **Example: The sizes of above matrices are 3×2, 1×4, 3×3, 2×1, and 1×1, respectively.**
  - **A matrix with only one column is called a *column matrix* (or a *column vector*), and a matrix with only one row is called a *row matrix* (or a *row vector*).**
  - **When discussing matrices, it is common to refer to numerical quantities as *scalars*.**

Matrices and Matrix Operations

- We often use capital letters to denote matrices and lowercase letters to denote numerical quantities.
- The entries that occurs in row *i* and column *j* of a matrix *A* will be denoted by a_{ij} or $(A)_{ij}$.
  - A general $m \times n$ matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

  - or $[a_{ij}]_{m \times n}$ or $[a_{ij}]$.
- Row and column matrices are denoted by boldface lowercase letters.

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Matrices and Matrix Operations

- **A matrix *A* with *n* rows and *n* columns is called a** *square matrix of order n*, **and entries a_{11}, a_{22},...,a_{nn} are said to be on the** *main diagonal* **of *A*.**

- **Operations on Matrices**
  - Definition**: Two matrices are defined to be** *equal* **if they have the same size and their corresponding entries are equal.**
  - Definition**: If *A* and *B* are matrices of the same size, then the** *sum* ***A*+*B* is the matrix obtained by adding the entries of *B* to the corresponding entries of *A*, and the** *difference* ***A*-*B* is the matrix obtained by subtracting the entries of *B* from the corresponding entries of *A*. Matrices of different sizes cannot be added or subtracted.**

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \text{ and } (A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Matrices and Matrix Operations

- Definition: **If *A* is any matrix and *c* is any scalar, then the** *product* ***cA* is the matrix obtained by multiplying each entry of the matrix *A* by *c*. The matrix *cA* is said to be a** *scalar multiple* **of *A*.**
- **If $A_1, A_2, \ldots, A_n$ are matrices of the same size and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form**

$$c_1A_1 + c_2A_2 + \cdots + c_nA_n$$

  is called a *linear combination* **of $A_1, A_2, \ldots, A_n$ with** *coefficients* **$c_1, c_2, \ldots, c_n$.**
- Definition: **If *A* is an $m \times r$ matrix and *B* is an $r \times n$ matrix, then the** *product* ***AB* is the $m \times n$ matrix whose entries are determined as follows. To find the entry in row *i* and column *j* of *AB*, single out row *i* from the matrix *A* and column *j* from the matrix *B*, Multiply the corresponding entries from the row and column together and then add up the resulting products.**

Matrices and Matrix Operations

– **Example: Multiplying Matrices**

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

– **The number of columns of the first factor *A* must be the same as the number of rows of the second factor *B* in order to form the product *AB*.**

- **Example: *A*: 3x4, *B*: 4x7, *C*: 7x3, then *AB*, *BC*, *CA* are defined, *AC*, *CB*, *BA* are *undefined*.**

Partitioned Matrices

– **A matrix can be subdivided or *partitioned* into smaller matrices by inserting horizontal and vertical rules between selected rows and columns.**

Matrices and Matrix Operations

- **Matrix Multiplication by Columns and by Rows**
  - **to find a particular row or column of a matrix product *AB***
    - ***j*th column matrix of *AB* = *A*[*j*th column matrix of *B*]**
    - ***i*th row matrix of *AB* = [*i*th row matrix of *A*]*B***
  - **Example: find the secoand column matrix of *AB***

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix},\qquad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

  - **If** a_1, a_2,...,a_m **denote the row matrices of *A* and** b_1, b_2,...,b_n **denote the column matrices of *B*, then**

$$AB=A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]=[A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n],\quad AB=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_m\end{bmatrix}B=\begin{bmatrix}\mathbf{a}_1B\\ \mathbf{a}_2B\\ \vdots\\ \mathbf{a}_mB\end{bmatrix}$$

Matrices and Matrix Operations

- **Matrix Products as Linear Combinations**
  - **Suppose that**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

  - **The product *A*x of a matrix *A* with a column matrix x is a linear combination of the column matrices of *A* with the coefficients coming from the matrix x.**

Matrices and Matrix Operations

– **The product y*A* of a 1×*m* matrix y with an *m*×*n* matrix *A* is a linear combination of the row matrices of *A* with scalar coefficients coming from y.**
  - **Example: Find *A*x and y*A*.**

$$A=\begin{bmatrix}-1 & 3 & 2\\ 1 & 2 & -3\\ 2 & 1 & -2\end{bmatrix}, \mathbf{x}=\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}, \mathbf{y}=\begin{bmatrix}1 & -9 & -3\end{bmatrix}$$

– **The *jth* column matrix of a product *AB* is a linear combination of the column matrices of *A* with the coefficients coming from the *jth* column of *B*.**
  - **Example:**

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}, B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

Matrices and Matrix Operations

- **Matrix Form of a Linear System**
  - **Consider any system of *m* linear equations in *n* unknowns.**

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

We can replace the m equations in the system by the single matrix equation

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

***A*x=b**

Matrices and Matrix Operations

- A is called the coefficient matrix.
- The augmented matrix of this system is

$$[A\,|\,\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- **Transpose of a Matrix**
  - Definition: **If *A* is any $m \times n$ matrix, then the** *transpose of* ***A*, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of *A*.**
    - **Example: Find the transposes of the following matrices.**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix},\quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix},\quad C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix},\quad D = \begin{bmatrix} 4 \end{bmatrix}$$

Matrices and Matrix Operations

– $(A^T)_{ij} = (A)_{ji}$
– Definition: **If *A* is a square matrix, then the** *trace of A*, **denoted by tr(*A*), is defined to be the sum of the entries on the main diagonal of *A*. The trace of *A* is undefined if *A* is not a square matrix.**
  - **Example: Find the traces of the following matrices.**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$Tr(A) = a_{11} + a_{22} + a_{33}$

Inverses: Rules of Matrix Arithmetic

- **Properties of Matrix Operations**
  - **Commutative law for scalars is not necessarily true.**
    1. **when *AB* is defined by *BA* is undefined**
    2. **when *AB* and *BA* have different sizes**
    3. **It is still possible that *AB* is not equal to *BA* even when 1 and 2 holds.**

  - **Example:**

$$A=\begin{bmatrix}-1 & 0\\ 2 & 3\end{bmatrix},\quad B=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$$

  - **Laws that still hold in matrix arithmetic**

(a) $A+B=B+A$ **(Commutative law for addition)**

(b) $A+(B+C)=(A+B)+C$ **(Associative law for addition)**

(c) $A(BC)=(AB)C$ **(Associative law for multiplication)**

(d) $A(B+C)=AB+AC$ **(Left distributive law)**

(e) $(B+C)A=BA+CA$ **(Right distributive law)**

(f) $A(B-C)=AB-AC$

(g) $(B-C)A=BA-CA$

(h) $a(B+C)=aB+aC$

(i) $a(B-C)=aB-aC$

(j) $(a+b)C=aC+bC$

(k) $(a-b)C=aC-bC$

(l) $a(bC)=(ab)C$

(m) $a(BC)=(aB)C=B(aC)$

Inverses: Rules of Matrix Arithmetic

- **Zero Matrices**
  - **A matrix, all of whose entries are zero, is called a** *zero matrix***.**
  - **A zero matrix is denoted by** ***0*** **or** $\boldsymbol{0_{m\times n}}$**. A zero matrix with one column is denoted by** 0**.**
  - **The cancellation law does not necessarily hold in matrix operation.**
  - **Valid rules in matrix arithmetic for zero matrix.**

    (a) A+0 = 0+A = A

    (b) A-A = 0

    (c) 0-A = -A

    (d) A0 = 0; 0A = 0
- **Identity Matrices**
  - **square matrices with 1's on the main diagonal and 0's off the main diagonal are called** *identity matrices* **and is denoted by** ***I***

Inverses: Rules of Matrix Arithmetic

- **Properties of Inverses**
  - Theorem: **If *B* and *C* are both inverses of the matrix *A*, then *B*=*C*.**
  - **If *A* is invertible, then its inverse will be denoted by A^{-1}.**
  - $AA^{-1}=I$ **and** $A^{-1}A=I$
  - Theorem: **The matrix**

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad-bc\neq 0$, in which case the inverse is given by the formula

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Inverses: Rules of Matrix Arithmetic

- Theorem**: If *A* and *B* are invertible matrices of the same size, then *AB* is invertible and** $(AB)^{-1} = B^{-1}A^{-1}$**.**
  - **Generalization: The product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverse in the reverse order.**
  - **Example: Inverse of a product**

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

- **Powers of a Matrix**
  - Definition**: If *A* is a square** matrix**, then we define the nonnegative integer powers of *A* to be**

$$A^0 = I, \qquad A^n = \underbrace{AA\cdots A}_{n\text{ factors}} \quad (n > 0)$$

Moreover, if *A* is invertible, then we define the nonnegative integer powers to be

$$A^{-n} = \left(A^{-1}\right)^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n\text{ factors}}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Inverses: Rules of Matrix Arithmetic

- Theorem: (Laws of Exponents) **If *A* is a square matrix and *r* and *s* are integers, then**

$$A^r A^s = A^{r+s}, \quad \left(A^r\right)^s = A^{rs}$$

- Theorem: **If *A* is an invertible matrix. then**
  - ***(a)*** **A^{-1} is invertible and $(A^{-1})^{-1}=A$**
  - ***(b)*** **A^n is invertible and $(A^n)^{-1}=(A^{-1})^n$ for $n = 0, 1, 2,...$**
  - **(c) For any nonzero scalar *k*, the matrix *kA* is invertible and**

$$\left(kA\right)^{-1} = \frac{1}{k} A^{-1}$$

  - **Example: Find A^3 and A^{-3}** $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

- **Polynomial Expressions Involving Matrices**
  - **If *A* is a $m \times m$ square matrix and if $p(x)=a_0+a_1x+...+a_nx^n$ is any polynomial, the we define $p(A) = a_0I+a_1A+...+a_nA^n$ where *I* is the $m \times m$ identity matrix.**

Inverses: Rules of Matrix Arithmetic

- **Properties of the Transpose**
  - Theorem: **If the sizes of the matrices are such that the stated operations can be performed, then**
    - **(a)** $((A)^T)^T = A$
    - **(b)** $(A+B)^T = A^T + B^T$ **and** $(A-B)^T = A^T - B^T$
    - **(c)** $(kA)^T = kA^T$, **where** k **is any scalar**
    - **(d)** $(AB)^T = B^T A^T$
  - **The transpose of a product of any number of matrices is equal to the product of their transpose in the reverse order.**
- **Invertibility of a Transpose**
  - Theorem: **If** A **is an invertible matrix, then** A^T **is also invertible and**

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Elementary Matrices and a Method for Finding A^{-1}

– Definition: **An $n \times n$ matrix is called an *elementary matrix* if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation.**

  - **Examples:**

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

– Theorem: **If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.**

  - **Example:**

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Elementary Matrices and a Method for Finding A^{-1}

- **Inverse row operations**

Row Operation on *I* That Produces *E*	*Row Operations on E That Reproduces I*
Multiply row *i* by *c*≠0	**Multiply row *i* by 1/*c***
Interchange row *i* and *j*	**Interchange row *i* and *j***
Add c times row *i* to row *j*	**Add –*c* times row *i* to row *j***

- Theorem: **Every elementary matrix is invertible, and the inverse is also an elementary matrix.**
- Theorem: **If *A* is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.**

  ***(a)* *A* is invertible.**

  ***(b)* *A*x=0 has only the trivial solution.**

  (c) The reduced row-echelon form of *A* is I_n.

  ***(d)* *A* is expressible as a product of elementary matrices.**

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply the second row by 7

Multiply the second row by $^1/_7$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Interchange the first and second rows

Interchange the first and second rows

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add 5 times the second row to the first

Add – 5 times the second row to the first

Elementary Matrices and a Method for Finding A^{-1}

- **Row Equivalence**
  - **Matrices that can be obtained from one another by a finite sequence of elementary row operations are said to be** *row equivalence*.
  - **An $n \times n$ matrix A is invertible if and only if it is row equivalent to the $n \times n$ identity matrix.**
- **A Method for Inverting Matrices**
  - **To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtained A^{-1}.**
  - **Example: Find the inverse of**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Sample Problem 7:

(a) **Determine the inverse for A, where:**

(b) **Use the inverse determined above to solve the system of linear equations:**

Example: Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

We added –2 times the first row to the second and –1 times the first row to the third

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

We added 2 times the second row to the third

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

We multiplied the third row by -1

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added 3 times the third row to the second and –3 times the third row to the first

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added –2 times the second row to the first

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Elementary Matrices and a Method for Finding A^{-1}

- If an $n \times n$ matrix is not invertible, then it cannot be reduced to I_n by elementary row operations, i.e. the reduced row-echelon form of *A* has at least one row of zeros.
- We may stop the computations and conclude that the given matrix is not invertible when the above situation occurs.
  - Example: Show that A is not invertible.

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 - 2R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Since we have obtained a row of zeros on the left side, A is not invertible.

Further Results on Systems of Equations and Invertibility

- **A Basic Theorem**
  - Theorem**: Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.**
- **Solving Linear Systems by Matrix Inversion**
  - **method besides Gaussian and Gauss-Jordan elimination exists**
  - Theorem**: If *A* is an invertible *n*×*n* matrix, then for each *n*×1 matrix** b**, the system of equations *A*x=b has exactly one solution, namely,** x**=*A*⁻¹**b**.**
  - **Example: Find the solution of the system of linear equations.**

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 3 \\ x_1 & & & + & 8x_3 & = & 17 \end{array}$$

  - **The method applies only when the system has as many equations as unknowns and the coefficient matrix is invertible.**

Solution:

*In matrix form the system can be written as A**x=b**, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

It is shown that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By theorem the solution of a system is

$$x = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad or \;\; x_1 = 1, x_2 = -1, x_3 = 2$$

Further Results on Systems of Equations and Invertibility

- **Linear Systems with a Common Coefficient Matrix**
  - **solving a sequence of systems $A\mathbf{x}=\mathbf{b}_1$, $A\mathbf{x}=\mathbf{b}_2$, $A\mathbf{x}=\mathbf{b}_3$,..., $A\mathbf{x}=\mathbf{b}_k$, each of which has the same coefficient matrix *A***
  - **If *A* is invertible, then the solutions can be obtained with one matrix inversion and *k* matrix multiplications.**
  - **A more efficient method is to form the matrix**

$$\left[A \,|\, \mathbf{b}_1 \,|\, \mathbf{b}_2 \,|\, \cdots \,|\, \mathbf{b}_k\right]$$

    By reducing the above augmented matrix to its reduced row-echelon form we can solve all *k* systems at once by Gauss-Jordan elimination.
    - **also applies when *A* is not invertible**
    - **Example: Solve the systems**

$$(a)\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 5 \\ x_1 & & & + & 8x_3 & = & 9 \end{array} \qquad (b)\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 6 \\ x_1 & & & + & 8x_3 & = & -6 \end{array}$$

Solution:

Two systems have the same coefficient matrix. If we augment this coefficient matrix with the column of constants on the right sides of these systems, we obtain

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right] \qquad \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

(a) $x_1 = 1, x_2 = 0, x_3 = 1$ *(b)* $x_1 = 2, x_2 = 1, x_3 = -1$

Further Results on Systems of Equations and Invertibility

- **Properties of Invertible Matrices**
  - Theorem: **Let A be a square matrix.**
    1. **If B is a square matrix satisfying $BA=I$, then $B=A^{-1}$.**
    2. **If B is a square matrix satisfying $AB=I$, then $B=A^{-1}$.**
  - Theorem: **If A is an $n \times n$ matrix, then the following are equivalent.**
    1. **A is invertible.**
    2. $Ax=0$ **has only the trivial solution.**
    3. **The reduced row-echelon form of A is I_n.**
    4. **A is expressible as a product of elementary matrices.**
    5. $Ax=b$ **is consistent for every $n \times 1$ matrix** b.
    6. $Ax=b$ **has exactly one solution for every $n \times 1$ matrix** b.
  - Theorem: **Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.**
  - A Fundamental Problem: **Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices** b **such that the system of equations** $Ax=b$ **is consistent.**

Further Results on Systems of Equations and Invertibility

- If A is an invertible matrix, for every mx1 matrix b, the linear system Ax=b has the unique solution x=A^{-1}b.
- If A is not square or not invertible, the matrix b must usually satisfy certain conditions in order for Ax=b to be consistent.
  - Example: Find the conditions in order for the following systems of equations to be consistent.

(a)
$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 + 8x_3 &= b_3 \end{aligned}$$

Diagonal, Triangular, and Symmetric Matrices

- **Diagonal Matrices**
  - **A square matrix in which all the entries off the main diagonal are zero are called** *diagonal matrices***.**
    - **Example:**

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

  - **A general** $n \times n$ **diagonal matrix** D **can be written as**

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

  - **A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero.**

Diagonal, Triangular, and Symmetric Matrices

- **Diagonal Matrices**
  - **A square matrix in which all the entries off the main diagonal are zero are called** *diagonal matrices*.
    - **Example:**

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

  - **A general** $n \times n$ **diagonal matrix** D **can be written as**

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

  - **A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero.**

Diagonal, Triangular, and Symmetric Matrices

– **Powers of diagonal matrices**

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

– **Example: Find *A*$^{-1}$, *A*^{5}, and *A*$^{-5}$ for** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

– **Multiplication of diagonal matrices**

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

Diagonal, Triangular, and Symmetric Matrices

- **Triangular Matrices**
  - **A square matrix in which all the entries above the main diagonal are zero is called** *lower triangular*, **and a square matrix in which all the entries below the main diagonal are zero is called** *upper triangular*. **A matrix that is either upper triangular or lower triangular is called** *triangular*.
  - **The diagonal matrices are both upper and lower triangular.**
  - **A square matrix in row-echelon form is upper triangular.**
  - **characteristics of triangular matrices**
    - **A square matrix $A=[a_{ij}]$ is upper triangular if and only if the ith row starts with at least i-1 zeros.**
    - **A square matrix $A=[a_{ij}]$ is lower triangular if and only if the jth column starts with at least j-1 zeros.**
    - **A square matrix $A=[a_{ij}]$ is upper triangular if and only if $a_{ij}=0$ for $i>j$.**
    - **A square matrix $A=[a_{ij}]$ is lower triangular if and only if $a_{ij}=0$ for $i<j$.**

Diagonal, Triangular, and Symmetric Matrices

– Theorem1.7.1:

(a) **The transpose of a lower triangular matrix is upper triangular, and the transpose of a upper triangular matrix is lower triangular.**

(b) **The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.**

(c) **A triangular matrix is invertible if and only if its diagonal entries are all nonzero.**

(d) **The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.**

– **Example: Let**

$$A=\begin{bmatrix}1 & 3 & -1\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix},\quad B=\begin{bmatrix}3 & -2 & 2\\ 0 & 0 & -1\\ 0 & 0 & 1\end{bmatrix}$$

Find A^{-1} and AB.

$$AB=\begin{bmatrix}3 & -2 & -2\\ 0 & 0 & 2\\ 0 & 0 & 5\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1 & \frac{-3}{2} & \frac{7}{5}\\ 0 & \frac{1}{2} & -\frac{2}{5}\\ 0 & 0 & \frac{1}{5}\end{bmatrix}$$

Diagonal, Triangular, and Symmetric Matrices

- **Symmetric Matrices**
  - **A square matrix is called** *symmetric* **if $A = A^T$.**
  - **Examples:** $\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$
  - **A matrix $A=[a_{ij}]$ is symmetric if and only if $a_{ij} = a_{ji}$ for all i, j.**
  - Theorem**: If A and B are symmetric matrices with the same size, and if k is any scalar, then:**

    (a) **A^T is symmetric.**

    (b) **$A+B$ and $A-B$ are symmetric.**

    (c) **kA is symmetric.**
  - **The product of two symmetric matrices is symmetric if and only if the matrices commute(i.e. AB=BA).**

Diagonal, Triangular, and Symmetric Matrices

- **In general, a symmetric matrix need not be invertible.**
- Theorem**: If *A* is an invertible symmetric matrix, then A^{-1} is symmetric.**

- **Products AA^T and A^TA**
  - **Both AA^T and A^TA are square matrices and are always symmetric.**
  - **Example:**

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

  - Theorem**: If *A* is invertible matrix, then AA^T and A^TA are also invertible.**